ECE 544: Pattern Recognition

A. G. Schwing

University of Illinois at Urbana-Champaign, 2021

Scribe & Exercises

Optimization Primal

Goals of this lecture

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- Understanding the basics of optimization

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Reading Material

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Reading Material

- S. Boyd and L. Vandenberghe; Convex Optimization; Chapters 2-4

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- Linear Regression

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Analytically computable optimum vs. gradient descent

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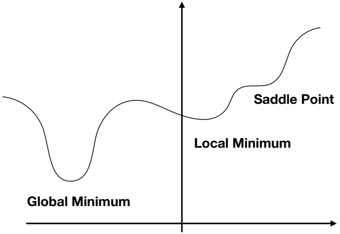

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https://www.strawpoll.me/19324883

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- Often compromise between accuracy and computation time

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- Convex program when all f_i convex (generalizes the above)

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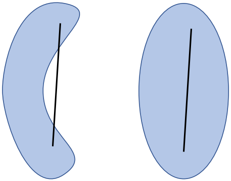

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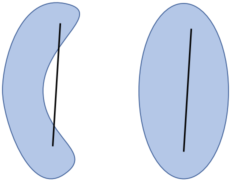

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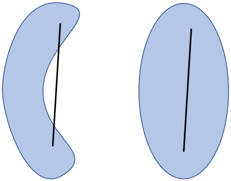

Example: Polyhedron

$$\{\boldsymbol{w} | \boldsymbol{A}\boldsymbol{w} \leq \boldsymbol{b}, \boldsymbol{C}\boldsymbol{w} = \boldsymbol{d}\}$$

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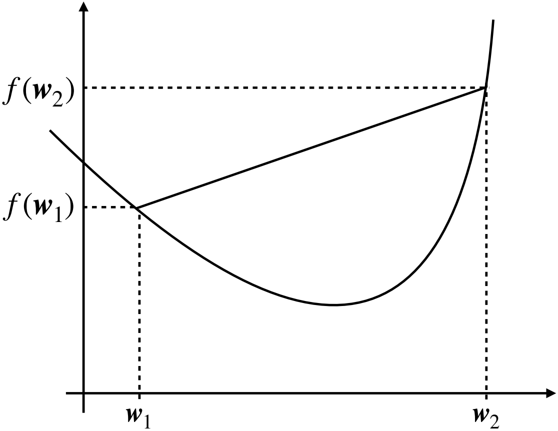

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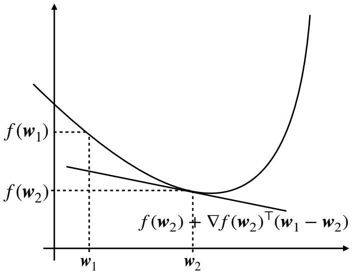

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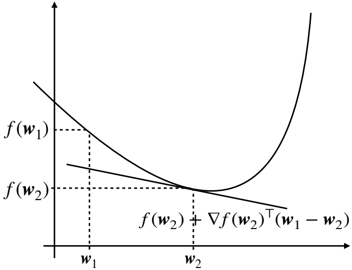

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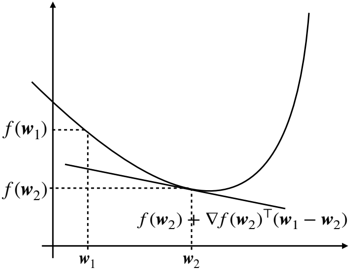

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- If f is twice differentiable, then f is convex if and only if its domain is convex and $\nabla^2 f(\boldsymbol{w}) \succeq 0\ \forall \boldsymbol{w}$ in the domain

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- Log-Sum-Exp: $\log(\exp(w_1) + \ldots + \exp(w_d))$

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Show that $\log(1 + \exp(x))$ is convex for $x \in \mathbb{R}$

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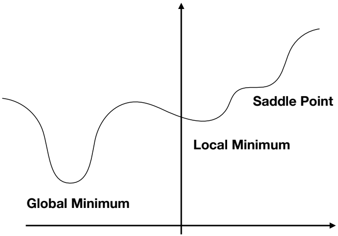

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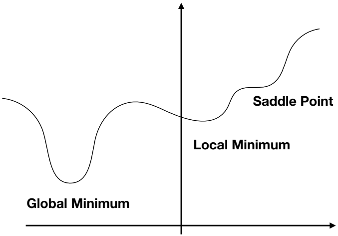

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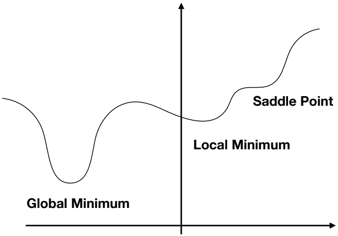

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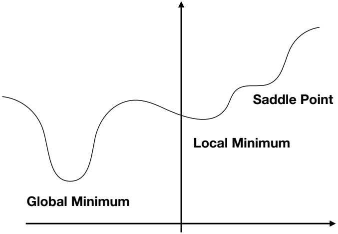

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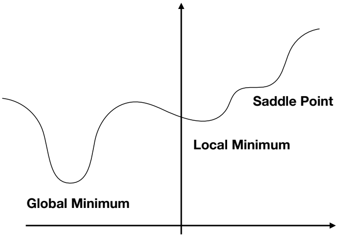

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This makes convex optimization special!

Algorithms to search for the optimum?

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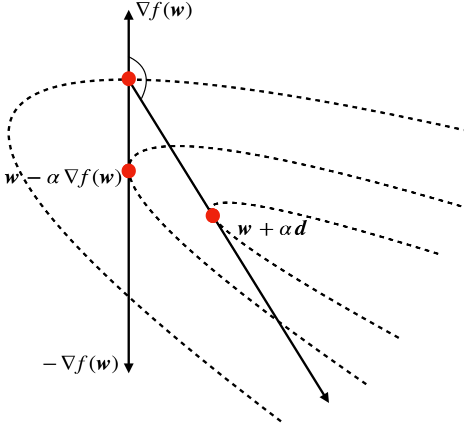

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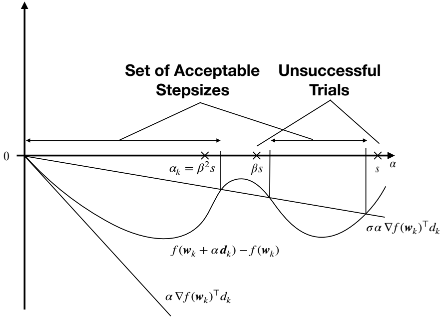

How long does it take to find the optimum?

Goal:
How many iterations k for

$$f(\boldsymbol{w}_k) - f(\boldsymbol{w}^*) \leq \epsilon$$

Two important properties:

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- Lipschitz continuous gradient

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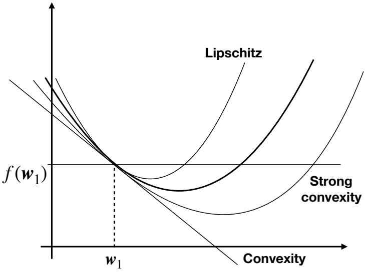

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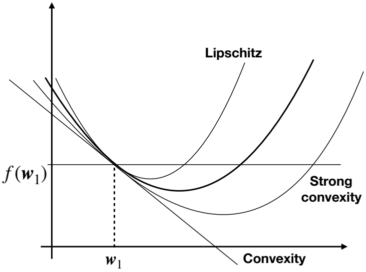

if f twice differentiable

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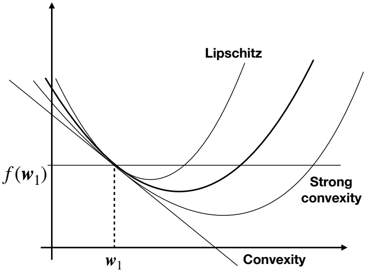

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$$\sigma I \prec \nabla^2 f(\boldsymbol{w}) \prec LI \qquad \forall \boldsymbol{w}$$

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How to pick $\alpha \boldsymbol{d}_k$? Minimize w.r.t. $\boldsymbol{w}_{k+1}$ right-hand-side of upper bound

$$f(\boldsymbol{w}_{k+1}) \leq f(\boldsymbol{w}_k) + \nabla f(\boldsymbol{w}_k)^\top (\boldsymbol{w}_{k+1} - \boldsymbol{w}_k) + \frac{L}{2} \|\boldsymbol{w}_{k+1} - \boldsymbol{w}_k\|_2^2$$

Hence

$$\begin{array}{rcll} \alpha \boldsymbol{d}_k & = & -\frac{1}{L} \nabla f(\boldsymbol{w}_k) & \\ f(\boldsymbol{w}_{k+1}) & \leq & f(\boldsymbol{w}_k) - \frac{1}{2L} \|\nabla f(\boldsymbol{w}_k)\|_2^2 & \text{Bound on guaranteed progress} \end{array}$$

Bound on sub-optimality from strong convexity:

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Guaranteed progress:

$$f(\boldsymbol{w}_{k+1}) \leq f(\boldsymbol{w}_k) - \frac{1}{2L}\|\nabla f(\boldsymbol{w}_k)\|_2^2$$

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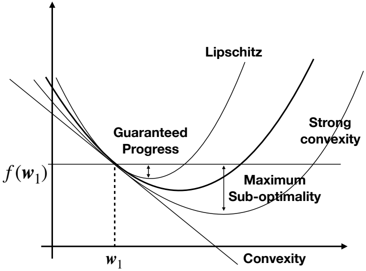

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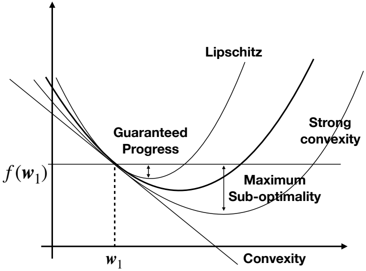

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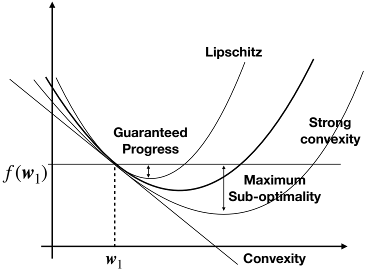

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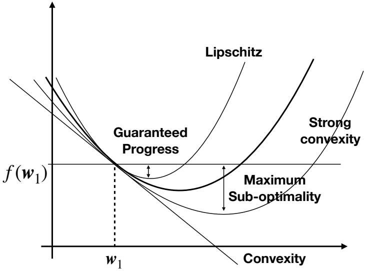

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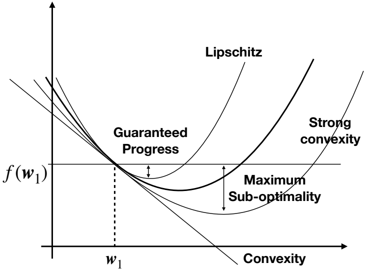

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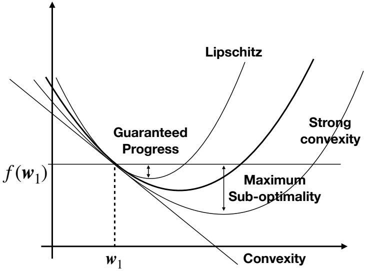

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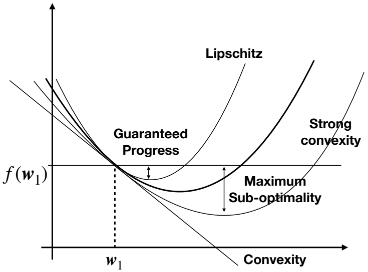

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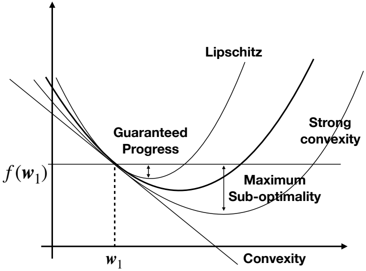

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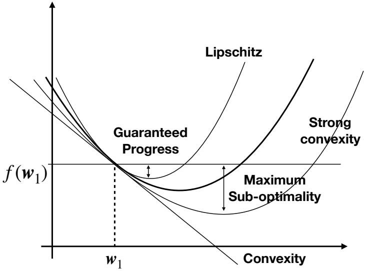

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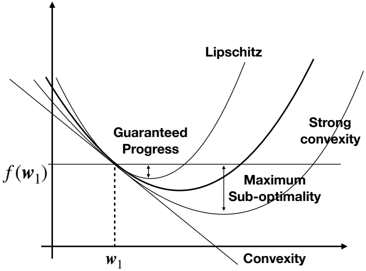

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Lipschitz bound and $\boldsymbol{w}_2 = \boldsymbol{w}_1 - \alpha \nabla f(\boldsymbol{w}_1)$ yields

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$f(\boldsymbol{w}_i)$ non-increasing:

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$$k \geq O(1/\epsilon)$$

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Are these rates optimal?

Gradient with momentum

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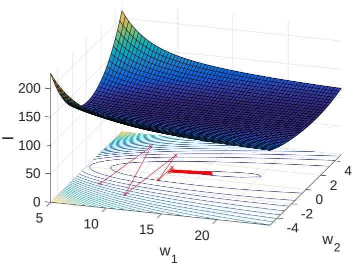
Intuition:



Gradient with momentum
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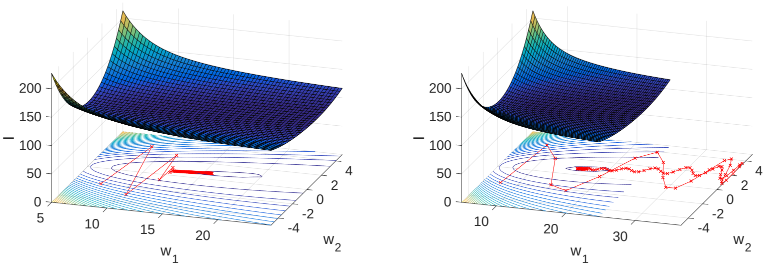

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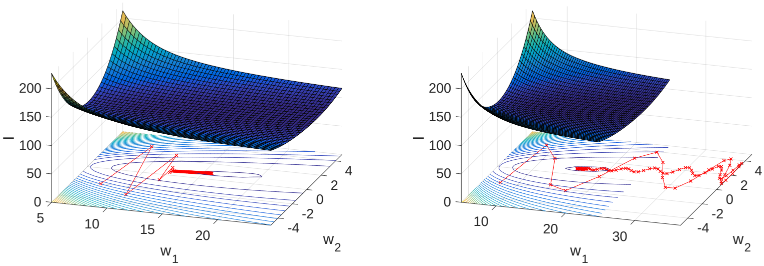

Video

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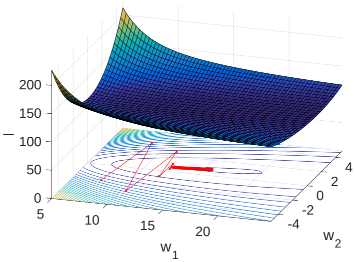

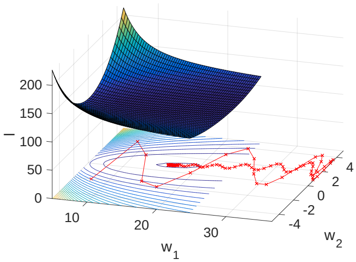

Video

- Polyak's method (aka heavy-ball)

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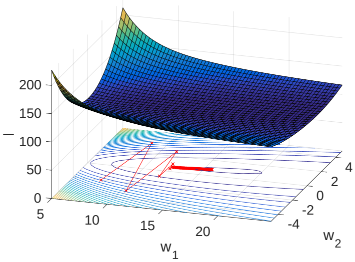

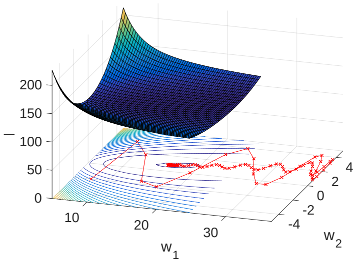

Video

- Polyak's method (aka heavy-ball)

$$\boldsymbol{w}_{k+1} = \boldsymbol{w}_k - \alpha_k \nabla f(\boldsymbol{w}_k) + \beta_k(\boldsymbol{w}_k - \boldsymbol{w}_{k-1})$$

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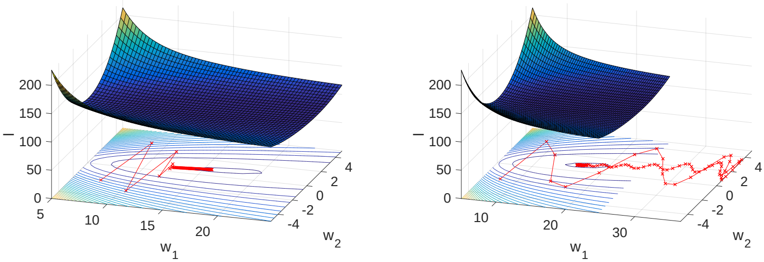

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- Momentum method in deep learning

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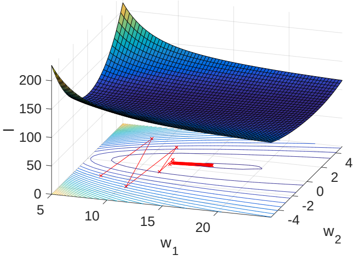

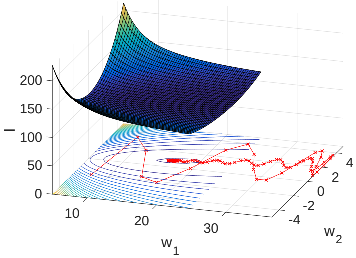

Video

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- Momentum method in deep learning

$$\begin{aligned} \mathbf{v}_{k+1} &= \beta \mathbf{v}_k + \nabla f(\mathbf{w}_k) \\ \mathbf{w}_{k+1} &= \mathbf{w}_k - \alpha \mathbf{v}_{k+1} \end{aligned}$$

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- Iteration complexity is linear in the number of samples $|\mathcal{D}|$
- A large dataset makes gradient computation slow

Recall the structure of our optimization problems:

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How to deal with this?

Stochastic gradient descent

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Convergence rates for stochastic gradient descent:

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Stochastic vs. deterministic (strongly convex)

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- Iteration complexity:

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Can we get the best of both worlds?

Many related algorithms:

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- SDCA (Shalev-Shwartz and Zhang 2013)
- SVRG (Johnnson and Zhang 2013)
- MISO (Mairal 2015)
- Finito (Defazio 2014)
- SAGA (Defazio, Bach, Lacoste-Julien 2014)
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Idea: variance reduction

Example: SVRG

- Initialize $\hat{\boldsymbol{w}}$
- For epoch 1, 2, 3, ...
  - Compute $\nabla f(\hat{\boldsymbol{w}}) = \sum_{i \in \mathcal{D}} \nabla \ell_i(\hat{\boldsymbol{w}})$
  - Initialize $\boldsymbol{w}_0 = \hat{\boldsymbol{w}}$
  - For t in length of epochs

$$\boldsymbol{w}_t = \boldsymbol{w}_{t-1} - \alpha \left[\nabla f(\hat{\boldsymbol{w}}) + \nabla \ell_{i(t)}(\boldsymbol{w}_{t-1}) - \nabla \ell_{i(t)}(\hat{\boldsymbol{w}}) \right]$$

  - Update $\hat{\boldsymbol{w}} = \boldsymbol{w}_t$
- Output $\hat{\boldsymbol{w}}$

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Up next:

- How to deal with constraints